Perturbation of parametrically excited solitary waves

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A direct perturbation analysis of solitary waves for a parametric Ginzburg Landau equation describing parametric excitation of waves in nonlinear dispersive and dissipative systems is presented. The method is used to study the influence on soliton dynamics of various perturbations, including external fields, stochastic driving forces, higher-order effects, and soliton interactions. A remarkable and quite general result of the analysis is that when the system is dissipative the dynamical motion induced by the perturbation is counteracted by the dissipative term, making dissipative solitary waves less sensitive to perturbations than solitons in the conservative case. [S1063-651X(97)04801-0]

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I. INTRODUCTION

Parametric excitation of solitary waves in dispersive, nonlinear, and dissipative systems is a topic of considerable interest in many area of physics. Examples include parametric excitation of surfaces waves in fluids [1–4], spin waves in ferromagnets [5–8], convection in binary mixtures [9] and in nematic liquid crystals [10], and parametric excitation of optical solitons [11–13]. In general, the existence of localized structures in these systems is due to the possibility of reducing the dynamical equations of motion which govern the physical problem to a universal equation for a complex order parameter $u=u(x,t)$. The simplest form of this equation is given by [2,3,6–8]

$$\partial_t u=(-\lambda+i\vartheta)u+\mu u^*+i\partial_x^2 u+iu^2u^*, \tag{1}$$

where $\lambda>0$ is the dissipation factor, ϑ is a detuning parameter, and $\mu>0$ is the parametric gain. Equation (1) is a parametric Ginzburg-Landau (PGL) equation which is known to possess localized, motionless solitary-wave solutions [2]. Previous studies on dynamical aspects of these localized structures were mainly restricted to the case of weak dissipation and weak parametric pumping [3,11], or to the dissipationless case [14], where Eq. (1) can be deduced from a Lagrangian density [15]. In the former case, the evolution equations for the solitary wave are derived by considering it as a perturbed soliton of the nonlinear Schrödinger (NLS) equation. Investigation of the dynamics of the NLS soliton under the action of dissipation and parametric excitation has demonstrated that, in contrast to the unperturbed case, in which the soliton's amplitude, phase and velocity may take arbitrary values, their stationary values are determined uniquely in the presence of these perturbations [3,11], and only the soliton position remains undetermined. This dynamical behavior is closely related to the symmetry breaking in the NLS equation induced by the parametric pumping [16].

Although this approach is capable of capturing the main mechanism which governs parametric excitation of solitary waves, a remarkable feature of Eq. (1) is that *exact* localized structures exist and they are *stable* in a wide region of the plane (λ,μ) without any assumption about the smallness of λ and μ [2,7]. The stability of these waves in the general case has been recently addressed in Ref. [7] by direct linear stability analysis of the PGL equation. For physical applications, a satisfactory theory of localized structures should also address the question of the influence that perturbing terms have on the soliton dynamics. These perturbing terms can represent fields externally imposed, or can describe higher-order corrections to the dynamical model expressed by Eq. (1) due to physical effects neglected at leading order; finally, they can arise from interaction with other localized structures as well. It is clear that a satisfactory perturbation theory of parametrically excited solitary waves may not be developed in general by considering Eq. (1) itself as a perturbed NLS equation, or, equivalently, by treating the solitary waves of the PGL equation as perturbed NLS solitons. The need for a direct perturbation theory of parametric solitons becomes particularly apparent when considering the strongly dissipative limit of Eq. (1), where dissipation becomes comparable to or larger than dispersion and nonlinearity of the system. Here we use the term soliton in a loose sense to indicate a localized solution of the underlying equation without necessarily assuming integrability of the equation. In this case, the phase, amplitude, and velocity of the unperturbed soliton are fixed, and it is expected that the effects of perturbations are to introduce a slow motion of the soliton position, which is the only degree of freedom allowed by the translational invariance of the unperturbed equation.

The aim of this paper is to introduce a direct perturbation theory to study the dynamics of parametrically excited solitary waves under the action of perturbations. In particular, the effects of external fields, stochastic perturbations, higher-order corrections to Eq. (1), and soliton interactions are discussed. The adiabatic evolution of the soliton position induced by the perturbation field is here derived as a solvability condition in a multiple scale expansion by using a direct perturbation approach [17–19]. It is shown that dissipation in the unperturbed equation is crucial in setting up the asymptotic expansion and may profoundly affect the soliton

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dynamics. In particular, it turns out that the typical temporal scale over which the soliton motion occurs is of order ϵ^{-2} if Eq. (1) is dissipative, and ϵ^{-1} for the conservative problem (i.e., for $\lambda=0$), where the smallness parameter ϵ^2 defines the order of magnitude of the perturbation. A remarkable physical implication thereof is that dissipative solitary waves are less sensitive to perturbations than conservative solitons. This feature can be best visualized by considering the weakly dissipative limit of Eq. (1), where the solitary wave behaves like a particle in an external field, and dissipation acts as a viscous force which counteracts the motion induced by the perturbation.

The paper is organized as follows. In Sec. II we review the basic properties of solitary waves for the PGL equation (1) which are needed to set up a perturbation theory. In Sec. III the dynamical equations of motion for the soliton position under the action of a generic perturbation are derived by using a multiple scale expansion method. Finally, in Sec. IV these equations are used to study the effects on soliton motion induced by particular perturbations which may be of interest for applications. In particular, effects of external driving fields, higher-order terms in Eq. (1), noise sources and soliton interactions are analyzed in detail. These examples allow us to illustrate the different reaction of dissipative versus conservative solitons to perturbations.

II. SOLITARY WAVES OF THE PGL EQUATION

The questions of existence and stability of solitary waves for the PGL equation (1) were investigated in previous papers [2,3,7]. In this section we review the basic results of these analyses which are needed to develop a perturbation theory. Localized solutions of Eq. (1) have the wave form [2]

$$u_{\pm}(x)=\sqrt{2}\beta_{\pm}\ \mathrm{sech}[\beta_{\pm}(x-\xi)]\exp(i\varphi) \tag{2}$$

where the phase φ and the amplitude $\beta_{\pm}$ of the solitary wave are given by

$$\cos(2\varphi)=\lambda/\mu, \quad \beta_{\pm}^2=-\vartheta\pm\sqrt{\mu^2-\lambda^2}, \tag{3}$$

and ξ is an arbitrary real constant parameter which defines the soliton position. Contrary to the soliton solutions of the NLS equation, which depend on four arbitrary real parameters (soliton position, phase, velocity and amplitude), a remarkable feature of the solitary wave (2) is that its amplitude, velocity and phase are fixed, and its position is the only allowed degree of freedom. This feature is closely related to the fact that the parametric term in Eq. (1) breaks three of the four symmetries which are typical of the NLS equation [3,16]. Strictly speaking, for a chosen value of β in Eq. (3), the phase of the solitary wave may assume two values which differ each other by π, so that two polarities can be associated to the solitary wave (2).

The domain of existence of solitary waves (2) is trivially determined by the condition $\beta_{\pm}>0$. Stability of these solutions against small perturbations is a crucial point which must be considered to derive a perturbation theory of the PGL equation. Some analytical and physical insights into the stability problem may be obtained by using an adiabatic approach based on the unperturbed NLS equation associated with Eq. (1) [3,11,16]. This method, however, has limited validity, as it cannot predict all classes of instability, and it assumes weak dissipation and weak parametric pumping. A more global stability analysis of solitary waves for the PGL equation has recently been given in Ref. [7] by using standard linear stability methods. Although this way a complete investigation of the stability problem can be done only numerically, this is not a serious limitation from the point of view of a perturbation theory of the PGL equation. In fact, it will be shown that the leading-order effects of the perturbation can be captured analytically with only the knowledge of the exact solitary-wave form (2). By setting $u=u_{\pm}+(\nu_1+i\nu_2)\exp(i\varphi)$ in Eq. (1), where ν_1 and ν_2 are small real perturbations, the linearized equations which govern the evolution of perturbations are given by

$$\partial_t\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}=\mathcal{L}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}. \tag{4}$$

The linear operator $\mathcal{L}$ in Eq. (4) is defined by

$$\mathcal{L}=\begin{pmatrix}\mathcal{L}_{11} & \mathcal{L}_{12}\\ \mathcal{L}_{21} & \mathcal{L}_{22}\end{pmatrix} \tag{5}$$

where

$$\mathcal{L}_{11}=0,$$

$$\mathcal{L}_{12}=-2\vartheta-\beta_{\pm}^2-\partial_x^2-2\beta_{\pm}^2\ \mathrm{sech}^2[\beta_{\pm}(x-\xi)],$$

$$\mathcal{L}_{21}=-\beta_{\pm}^2+\partial_x^2+6\beta_{\pm}^2\ \mathrm{sech}^2[\beta_{\pm}(x-\xi)],$$

$$\mathcal{L}_{22}=-2\lambda.$$

The solitary wave given by Eqs. (2) and (3) is linearly stable provided that $\mathrm{Re}(\sigma)\leqslant 0$ for each eigenvalue σ of the linear operator $\mathcal{L}$. The continuous eigenvalue spectrum of $\mathcal{L}$ can be easily calculated by considering the asymptotical behavior of Eq. (4) as $x\to\infty$, and is given by

$$\sigma_{\pm}(\Omega)=-\lambda\pm\sqrt{\mu^2-(\Omega^2-\vartheta)^2}, \tag{6}$$

where Ω is an arbitrary real parameter. The eigenfunctions associated with these eigenvalues are the radiation modes whose asymptotical behavior as $x\to\infty$ is $(\nu_1,\nu_2)\propto\cos(\Omega x)$ or $(\nu_1,\nu_2)\propto\sin(\Omega x)$. From Eq. (6) it follows that the solitary wave (2) is always unstable when $\vartheta>0$. For $\vartheta<0$, stability against the growth of radiation modes is ensured provided that the parametric pumping μ is less than $\sqrt{\lambda^2+\vartheta^2}$. The discrete eigenvalue spectrum of the operator $\mathcal{L}$ may be computed for general parameter values only numerically [7]. Two general properties of the discrete eigenvalue spectrum may, however, be stated: (1) $\sigma=0$ belongs to the discrete spectrum, i.e., the operator $\mathcal{L}$ is singular; and (2) for the solution u_- there always exists an eigenvalue with a positive real part. The first property is merely a consequence of the translational invariance of Eq. (1), so that the translation mode $\partial_x u_{\pm}$ is always an eigenvector of $\mathcal{L}$ corresponding to the zero eigenvalue. The second property can be demonstrated by using the maximum principle for positive definite operators [7], and shows that the solitary wave u_- is always linearly unstable, in agreement with the predictions of the adiabatic method [3]. Therefore, in the following, we will consider only the solution u_+ corresponding to the upper sign in Eq. (3). The stability of this solution in the plane (λ,μ) is hence contained between the lines $\mu=\lambda$ and $\mu=\sqrt{\lambda^2+\vartheta^2}$ which define the boundaries of existence and of stability with respect to radiation modes of solitary waves, respectively. These are in fact the exact stability boundaries predicted by the adiabatic analysis [3]. Direct numerical computation of the discrete eigenvalue spectrum shows indeed that, for a wide range of parameters λ and μ inside this domain, all other discrete eigenvalues of the operator $\mathcal{L}$ have a negative real part [7]. However, a comparison of the stability domains as obtained from the adiabatic analysis and from direct linear stability analysis shows that an instability due to the emergence of a discrete eigenvalue with a positive real part may arise for weak dissipation and for strong parametric pumping [7]. The effects of this instability on the soliton dynamics were numerically investigated in Ref. [20]. In this paper we will consider only parameter values where such instability is absent and the solitary wave (2) is linearly stable. This implies, in particular, $\vartheta<0$ and $\lambda<\mu<\sqrt{\lambda^2+\vartheta^2}$. As a final remark, it should be noted that in the dissipationless case, i.e., for $\lambda=0$, the system described by Eq. (1) is conservative, as it can be derived from a Lagrangian density [14,15]. In this case the eigenvalues (6) of the continuous spectrum lie on the imaginary axis, indicating that the solitary wave (2) is marginally stable with respect to radiation modes.

III. EVOLUTION EQUATIONS FOR THE PERTURBED SYSTEM

In this section the evolution equations for the solitary wave under the influence of a perturbation are derived as solvability conditions in a multiple scale perturbation expansion analysis. This method is quite standard, and it consists of assuming as the solution at leading order in the perturbation expansion a solitary wave of the unperturbed system whose free parameters are allowed to vary slowly in time. The slow time evolution of the soliton parameters is then obtained by eliminating secular terms arising in the perturbation expansion [3,17,18,21]. The starting point of the analysis is provided by Eq. (1) with a perturbation term

$$\partial_t u=(-\lambda+i\vartheta)u+\mu u^*+i\partial_x^2 u+iu^2u^*+\epsilon^2 P[u], \quad (7)$$

where ϵ^2 is a small parameter which defines the order of magnitude of the perturbation $P[u]$. The problem is to construct an asymptotic approximation of the perturbed solution $u=u(x,t;\epsilon)$ as $\epsilon\to 0$ which is valid uniformly in time. Therefore, we seek a perturbation expansion of u in the form

$$u=[u^{(0)}+\epsilon u^{(1)}+\epsilon^2 u^{(2)}+\cdots]\exp(i\varphi_0), \quad (8)$$

where the constant phase factor φ_0 in Eq. (8) will be defined later, and we require that the asymptotic expansion (8) be uniformly valid in time. This condition can be generally satisfied assuming the unperturbed solitary wave (2) as a solution at leading order in the expansion (8) provided that the soliton position ξ is allowed to vary slowly in time. This degree of freedom is fundamental in order to remove secular terms that arise in the perturbation expansion. To this aim, we introduce a multiple scale for time; that is, we assume that the order parameter u depends on $T_0, T_1, T_2,\ldots$, where

$$T_0=t,\ \ T_1=\epsilon t,\ \ T_2=\epsilon^2 t,\ldots\ . \quad (9)$$

We note explicitly that the ansatz (8) and (9) allows the correction to the unperturbed solitary wave to be of lower order $[O(\epsilon)]$ than the perturbing term [which is of $O(\epsilon^2)$], and the soliton position to vary on a shorter time scale $(\sim\epsilon^{-1})$ than that which is naturally associated with the perturbation (i.e., $\sim\epsilon^{-2}$). The justification for this choice will be clear later. Essentially, it is related to the possibility of satisfying the secularity conditions when the PGL equation (1) becomes conservative. Furthermore, as we want to study the influence of perturbations both in dissipative $(\lambda\neq 0)$ and in conservative $(\lambda=0)$ cases, it is useful to set $\lambda=\lambda_0+\epsilon\lambda_1$. Hence the dissipative case corresponds to $\lambda_0\neq 0$, while $\lambda_0=0$ and $\lambda_1\neq 0$ correspond to what we will call the weakly dissipative limit. Note that in this limit dissipation is weak, but of lower order than the perturbation $\epsilon^2 P$. In order to distinguish the weakly dissipative limit from the genuine dissipative case, we will call the latter one the strongly dissipative limit. Finally, the conservative case is recovered from the weakly dissipative limit by further setting $\lambda_1=0$; any eventual dissipative term of order ϵ^2 can be included into the perturbation P.

Introducing ansatz (8) into Eq. (7), using the derivative rule $\partial_t=\partial_{T_0}+\epsilon\partial_{T_1}+\epsilon^2\partial_{T_2}+\cdots$ and collecting the terms of the same order in the equation so obtained, a hierarchy of equations for successive corrections to u is obtained. If the phase φ_0 in Eq. (7) is chosen in such a way that $\cos(2\varphi_0)=\lambda_0/\mu$, it follows that the solitary wave solution $u^{(0)}$ at leading order is given by

$$u^{(0)}(x)=\sqrt{2}\beta\delta\ \mathrm{sech}[\beta(x-\xi)], \quad (10)$$

where $\beta=[-\vartheta+(\mu^2-\lambda_0^2)^{1/2}]^{1/2}$, $\delta=\pm 1$, and the soliton position ξ is an arbitrary function of the slow time variables $T_1, T_2,\ldots$, but is constant with respect to T_0, i.e., $\partial_{T_0}\xi=0$. Note that, in writing Eq. (10), we have explicitly shown the two allowed values of the solitary wave phase by introducing the parameter δ. The equations at higher orders may be cast in the form

$$\partial_{T_0}u^{(k)}-\mathcal{L}u^{(k)}=G^{(k)} \quad (11)$$

$(k=1,2,3,\ldots)$, where the linear operator $\mathcal{L}$ is given by Eq. (5) with $\lambda=\lambda_0$, and $G^{(k)}$ depends only on functions of previous approximations. In particular, one has

$$G^{(1)}=-\lambda_1 u^{(0)}-\partial_{T_1}u^{(0)}, \quad (12)$$

$$G^{(2)}=-\lambda_1 u^{(1)}-\partial_{T_1}u^{(1)}-\partial_{T_2}u^{(0)}+iu^{(0)}u^{(1)}[2u^{(1)*}+u^{(1)}] + P[u^{(0)}\ \exp(i\varphi_0)]\exp(-i\varphi_0). \quad (13)$$

In general, because the operator $\mathcal{L}$ is singular, the solution $u^{(k)}$ of Eq. (11) will not be a limited function of T_0 unless the driving term $G^{(k)}$ in the equation is orthogonal to the null space of the adjoint operator $\mathcal{L}^+$. Thus, to avoid secular terms in Eq. (11), which prevent the solution $u^{(k)}$ from being bounded in time, the following solvability conditions must be satisfied:

$$(G^{(k)},w)=0, \tag{14}$$

where w is the singular eigenvector of the adjoint operator $\mathcal{L}^{+}$, and the brackets indicate the inner product defined by

$$(A,B)=\mathrm{Re}\left[\int_{-\infty}^{+\infty}A^{*}(x)B(x)dx\right].$$

This corresponds to the usual scalar product in $\mathcal{R}^2$ once the complex quantities are separated into real and imaginary parts. Hence the solvability conditions (14) allow one to determine the evolution equation of the soliton position ξ on various slow time scales. The singular eigenvector w of the adjoint operator $\mathcal{L}^{+}$ is computed in Appendix A, and is given by

$$w(x)=-2\lambda_0 f[\beta(x-\xi)]+i\partial_x u^{(0)} \tag{15}$$

where the explicit expression of the real function $f(y)$ is given by Eqs. (A18) and (A19) in the same appendix. It turns out that f is an odd function of $(x-\xi)$, has only one zero at $x=\xi$, and assumes opposite sign as $\partial_x u^{(0)}$. To further proceed in the analysis, the two cases $\lambda_0\neq 0$ and $\lambda_0=0$ must be distinguished.

A. Strongly dissipative limit ($\lambda_0\neq 0$)

In this case, the solvability condition at $O(\epsilon)$ yields

$$\int_{-\infty}^{+\infty}(\partial_{T_1}\xi\partial_x u^{(0)}-\lambda_1 u^{(0)})f\ dx=0, \tag{16}$$

where we used the relation $\partial_{T_1}u^{(0)}=\partial_\xi u^{(0)}\partial_{T_1}\xi=-\partial_x u^{(0)}\partial_{T_1}\xi$. Because f and $\partial_x u^{(0)}$ are odd functions of $(x-\xi)$, and $u^{(0)}$ is an even function, Eq. (16) yields

$$\partial_{T_1}\xi=0, \tag{17}$$

i.e., the soliton position does not vary on the time scale T_1. The solution at this order, $u^{(1)}=u_1^{(1)}+iu_2^{(1)}$, then satisfies the following equations:

$$\partial_{T_0}u_1^{(1)}-\mathcal{L}_{12}u_2^{(1)}=-\lambda_1 u^{(0)},$$

$$\partial_{T_0}u_2^{(1)}-\mathcal{L}_{21}u_1^{(1)}+2\lambda_0 u_2^{(1)}=0.$$

It is straightforward to show that the former equation can be satisfied by setting $u_2^{(1)}=-(\lambda_1/2)(\mu^2-\lambda_0^2)^{-1/2}u^{(0)}$, and that the solution $u_1^{(1)}$ of the latter equation may be chosen to be an even function of its argument. Applying the solvability condition at $O(\epsilon^2)$, after observing that for trivial symmetry properties the terms $-\lambda_1 u^{(1)}$ and $iu^{(0)}[2u^{(1)}u^{(1)*}+u^{(1)2}]$ of $G^{(2)}$ are orthogonal to w, we obtain the following evolution equation for the soliton position ξ on the time scale T_2:

$$\beta m\partial_{T_2}\xi=\mathrm{Re}\left[\int_{-\infty}^{+\infty}fP\ \exp(-i\varphi_0)dx\right]-\frac{1}{2\lambda_0}\mathrm{Im}\left[\int_{-\infty}^{+\infty}\partial_x u^{(0)}P\ \exp(-i\varphi_0)dx\right], \tag{18}$$

where the parameter $m>0$ is defined by

$$m=-\frac{1}{\beta}\int_{-\infty}^{+\infty}f\partial_x u^{(0)}dx, \tag{19}$$

and the perturbation P is intended to be calculated for $u=u^{(0)}\exp(i\varphi_0)$. The total time derivative of the soliton position is given by $\partial_t\xi=\epsilon\partial_{T_1}\xi+\epsilon^2\partial_{T_2}\xi+\cdots$. Then combining Eqs. (17) and (18), we finally obtain the following time evolution equation:

$$\beta m\partial_t\xi=\epsilon^2\ \mathrm{Re}\left[\int_{-\infty}^{+\infty}fP\ \exp(-i\varphi_0)dx\right]-\frac{\epsilon^2}{2\lambda_0}\mathrm{Im}\left[\int_{-\infty}^{+\infty}\partial_x u^{(0)}P\ \exp(-i\varphi_0)dx\right], \tag{20}$$

which is valid up to the long-time scale $\sim\epsilon^{-2}$. It should be noted that the constant m defined by Eq. (19), which gives a measure of the soliton ''inertia'' to the applied perturbation, is a function only of the parameter q given by Eq. (A6), which may vary in the interval $0<q<1$. The limits $q\to 0$ and $q\to 1$ correspond to $\mu\to(\lambda_0^2+\vartheta^2)^{1/2}$ and $\mu\to\lambda$, respectively, i.e., to the two stability boundaries of the solitary wave. Numerical computation of m by use of Eqs. (A18) and (A19) shows that for $q\to 0$ the soliton mass m diverges together with the function f, while the ratio f/m remains bounded. In this limit, the last term on the right-hand side of Eq. (20) may be neglected. Conversely, it turns out that the soliton mass remains limited and approaches the value $m=1$ in the opposite limit $q\to 1$.

B. Weakly dissipative limit ($\lambda_0=0$)

In this case, the solvability condition at $O(\epsilon)$ is always satisfied, and the solution at this order is given by

$$u^{(1)}=i\left[f\partial_{T_1}\xi-\frac{\lambda_1}{2\mu}u^{(0)}\right]. \tag{21}$$

This fact is closely related to the conservative nature of the unperturbed PGL equation in the weakly dissipative limit, where the linear operator $i\mathcal{L}$ becomes self-adjoint, and the driving term $G^{(1)}$ associated with the slow soliton motion is always orthogonal to the singular eigenvector of $\mathcal{L}$ [17]. Hence the evolution equation of the soliton position on the time scale T_1 must be determined by imposing the solvability condition at $O(\epsilon^2)$. The derivation of the solvability condition at order ϵ^2 is rather involved, and details of the calculations are given in Appendix B. It turns out that the evolution equation of the soliton position on the time scale T_1 is given by

$$\beta m \partial_{T_1}^2 \xi + 2\lambda_1 \beta m \partial_{T_1} \xi + \mathrm{Im}\left[\int_{-\infty}^{+\infty} \partial_x u^{(0)} P \exp(-i\varphi_0) dx\right] = 0. \tag{22}$$

Using the derivative rule $\partial_t = \epsilon \partial_{T_1} + \epsilon^2 \partial_{T_2} + \cdots$, and observing that $\lambda = \epsilon\lambda_1$, from Eq. (22) we finally obtain

$$\beta m \partial_t^2 \xi + 2\lambda \beta m \partial_t \xi + \epsilon^2 \mathrm{Im}\left[\int_{-\infty}^{+\infty} \partial_x u^{(0)} P \exp(-i\varphi_0) dx\right] = 0 \tag{23}$$

which is valid up to the time scale $t \sim \epsilon^{-1}$. In particular, for $\lambda = 0$, Eq. (23) describes the soliton dynamics for dissipationless systems.

Equations (20) and (23) represent the main results of this section, and allow one to study the effects of a generic perturbation on the soliton dynamics. Before discussing some particular forms of the perturbation P, which will be the subject of Sec. IV, a few general comments on the results obtained and their limits of validity are in order. Contrary to the NLS equation, the PGL equation (1) has only translational symmetry, so that its solitary wave (2) has only one family parameter, the soliton position ξ. The effect of perturbations on the solitary wave is merely to change the soliton position on a slow time scale. According to the dynamical equations (20) and (23), the typical time scale over which the soliton motion occurs is $\sim\epsilon^{-2}$ if the unperturbed PGL equation is dissipative, and $\sim\epsilon^{-1}$ in the conservative case. This means that *dissipative* solitary waves react to the applied perturbing field on a longer time scale than *conservative* solitary waves. Hence we may conclude that dissipation in the *unperturbed* system plays a fundamental role in reducing the motion induced by perturbations.

This rule is rather general, but perturbations exist for which such a rule is not valid. In fact, any perturbation P such that $P[u^{(0)} \exp(i\varphi_0)]\exp(-i\varphi_0)$ is a real function induces a slow motion of dissipative solitons on the time scale $\sim\epsilon^{-2}$ provided that the first integral on the right-hand side in Eq. (20) does not vanish. On the contrary, for weakly dissipative or conservative systems, from Eq. (22) it follows that $\partial_{T_1}\xi = 0$, i.e., the perturbation does not induce a motion of the soliton on the time scale ϵ^{-1}. Since Eq. (23) is only valid up to the time scale ϵ^{-1}, it is not capable of describing the slow soliton motion on longer time scales, and solvability conditions at higher orders should be considered. In particular, one can find a general class of perturbations which do not induce any soliton motion if the system is conservative, while they cause a soliton motion if the system is dissipative. This class of perturbations is discussed in Appendix C, where it is shown that a regular asymptotic expansion of the solution of Eq. (7) can be constructed in the conservative case. In other words, the secularity conditions at each order can be satisfied for these perturbations without requiring any slow-time dependence of the soliton position ξ. This means that the effect of the perturbation on the unperturbed soliton is merely to introduce a static modification of its shape. As a final remark, we note that in dissipative systems a weak perturbation does not excite radiation modes because the continuous eigenvalue spectrum of Eq. (4) has a negative real part. This is obvious in the strongly dissipative limit, where radiation modes are damped on the time scale T_0. In the weakly dissipative limit, this result is still valid. In fact, in this case the damping rate of the radiation modes is small (of order $\sim\epsilon$), but it is of lower order than the perturbing term ($\sim\epsilon^2$); that excludes excitation of radiation modes. Conversely, in the conservative case the continuous spectrum of the linear operator $\mathcal{L}$ lies on the imaginary axis, and the solitary wave is marginally stable with respect to the growth of radiation modes in the unperturbed problem. When the perturbation field is considered, secondary secularities (usually called resonances [18]) might arise whenever the driving terms in Eq. (11) are resonant with radiation modes. The study of these resonances is a nontrivial matter, and would require the knowledge of the continuous part of the Green function associated with the operator $\mathcal{L}$ [18]. This analysis, however, goes beyond the purpose of this work and will not be done here.

IV. ANALYSIS OF SOME PARTICULAR PERTURBATIONS

In this section we specialize the dynamical equations (20) and (23) to study the effects of some particular perturbations on the soliton dynamics. These examples allow us to show the different behavior of dissipative and conservative solitary waves under the action of a perturbation.

(1) *External driving force.* As a first simple example of perturbation, let us consider an external applied force, which we assume to be independent of u and dependent only upon x; i.e., let us assume

$$\epsilon^2 P = \epsilon^2 F(x). \tag{24}$$

We further assume that $F(x)$ is a real function of x and varies slowly on the scale of order $\sim\beta^{-1}$. In this case, the equation describing the soliton motion in the strongly dissipative limit becomes

$$\beta m \partial_t \xi = \epsilon^2 D \left(\frac{\partial F}{\partial x}\right)_\xi, \tag{25}$$

where

$$D = \frac{\cos(\varphi_0)}{\beta^2} \int_{-\infty}^{+\infty} y f(y) dy + \frac{\sin(\varphi_0)}{2\lambda_0} \sqrt{2}\pi\delta$$

is a constant parameter independent of ξ. From Eq. (25) it follows that the stationary fixed points of motion satisfy the equation $(\partial F/\partial x)_\xi = 0$; these equilibria are then stable states provided that $D(\partial^2 F/\partial x^2)_\xi < 0$. In the weakly dissipative limit, Eq. (23) yields instead

$$m \partial_t^2 \xi + 2\lambda m \partial_t \xi + D\epsilon^2 \left(\frac{\partial F}{\partial x}\right)_\xi = 0, \tag{26}$$

where now

$$D = -\frac{\sin(\varphi_0)}{\beta}\int_{-\infty}^{+\infty} x\,\partial_x u^{(0)} dx = \frac{\sqrt{2}\pi\delta\,\sin(\varphi_0)}{\beta}.$$

Equation (26) indicates that in this case the soliton behaves like a particle of mass m in the potential $U(\xi)=\epsilon^2 DF(\xi)$ under the action of a viscous force which is proportional to the dissipation coefficient λ of the system. The stable equilibrium points of motion are hence represented by the minima of the potential. Note that, due to the presence of the viscous force, the motion induced by the external potential is counteracted, and an equilibrium point is asymptotically reached. In particular, the transient stage toward the equilibrium point ξ is oscillatory if $\lambda^2<(1/m)(\partial^2 U/\partial x^2)_{\xi}$, and monothonic in the opposite case. In the conservative case ($\lambda=0$), the viscous force in Eq. (26) disappears, and oscillations of the soliton position around the equilibrium points are allowed.

(2) *Higher-order terms.* The dynamical model expressed by the PGL equation (1) is usually obtained by neglecting some higher-order terms which generally describe physical effects of a small entity. Among these effects, here we consider the delay in the nonlinear response of the system and higher-order dispersion. These perturbations are of particular interest in the optical context, where their effects may profoundly affect soliton propagation in long distance soliton transmission systems [22]. The noninstantaneous response of the nonlinearity gives rise to the following perturbing term:

$$\epsilon^2 P = -i\epsilon^2 u\,\partial_x|u|^2. \tag{27}$$

In the optical context, this perturbation describes nonlinear dissipation generated by the induced Raman scattering, and is responsible for a frequency downshift of optical solitons [22]. In the strongly dissipative limit, the equation for the soliton position under the action of the perturbation (27) becomes

$$m\,\partial_t\xi = \frac{S}{2\lambda_0}, \tag{28}$$

where the constant parameter S is given by

$$S = \frac{2\epsilon^2}{\beta}\int_{-\infty}^{+\infty}[u^{(0)}\partial_x u^{(0)}]^2 dx = \frac{32}{15}\,\epsilon^2\beta^4. \tag{29}$$

Hence the effect of delay in the nonlinear response is to induce a drift of the soliton position with a constant velocity $v=16\epsilon^2\beta^4/15\lambda_0 m$ which is of order ϵ^2. In the weakly dissipative limit, Eq. (23) yields instead

$$m\partial_t^2\xi + 2\lambda m\,\partial_t\xi - S = 0, \tag{30}$$

where S is given by Eq. (29). Therefore, if $\lambda\neq 0$, the effect of the perturbation is the same as in the strongly dissipative limit, except that now the constant drift velocity is of order ϵ. In the conservative case, which is obtained by setting $\lambda=0$ in Eq. (30), the soliton motion is different, and is characterized by a continuous acceleration. This result is analogous to that predicted for conservative solitons in the NLS equation, where the acceleration of the soliton is also associated with a continuous frequency downshift [22]. Hence the acceleration of the solitary wave stabilizes when the system is dissipative, and it travels with a small constant velocity. The other perturbation we briefly discuss is related to third-order dispersion, and is given by

$$\epsilon^2 P = \epsilon^2\partial_x^3 u. \tag{31}$$

In the strongly dissipative limit, the effect of the perturbation (31) is to induce a constant drift of the soliton position of order ϵ^2 in a similar way as for the perturbation (27). In the weakly dissipative limit, from Eqs. (22) and (31) it follows that the driving term vanishes and $\partial_{T_1}\xi=0$. In other words, in this case the perturbation (31) does not induce a soliton motion on the slow time scale $\sim\epsilon^{-1}$, and solvability conditions at higher orders should be considered. We note that perturbation (31) belongs to the general class of perturbations discussed in Appendix C, and therefore in the conservative case third-order dispersion does not induce a drift of the soliton.

(3) *Stochastic perturbations.* When the physical system is subjected to noise sources, the soliton position ξ may undergo a stochastic motion due to the translational invariance of Eq. (1). This effect is well known for the NLS equation and, in nonlinear optics, it is known as the Gordon-Haus effect [23]. The effect of noise on the soliton dynamics can be captured by assuming a perturbation of the form

$$\epsilon^2 P = \epsilon^2\sigma(x,t), \tag{32}$$

where $\sigma=\sigma_1+i\sigma_2$ is an appropriate complex stochastic variable whose statistical properties depend upon the physical nature of noise. A simple case is that where the stochastic field σ can be represented by two real independent Gaussian stochastic functions σ_1 and σ_2 of zero mean value with correlations defined as

$$\langle\sigma_1(x,t)\sigma_2(x',t')\rangle = 0,$$

$$\langle\sigma_1(x,t)\sigma_1(x',t')\rangle = D_1\delta(x-x')\delta(t-t'),$$

$$\langle\sigma_2(x,t)\sigma_2(x',t')\rangle = D_2\delta(x-x')\delta(t-t'). \tag{33}$$

For instance, in the context of optical solitons, stochastic sources with correlations given by Eq. (33) may describe quantum fluctuations as well as phase noise (such as that due to guided-acoustic-wave Brillouin scattering) [24]. Setting Eq. (32) into Eq. (20) and using the correlations given by Eq. (33), we find that the equation of motion for the soliton position in the strongly dissipative limit is described by the Langevin equation

$$\beta m\,\partial_t\xi = S(t), \tag{34}$$

where the zero mean value, real stochastic term $S(t)$ is δ correlated with the diffusion constant

$$D = \epsilon^4 D_1\int_{-\infty}^{+\infty}\left[f\,\cos(\varphi_0) + \frac{1}{2\lambda_0}\sin(\varphi_0)\partial_x u^{(0)}\right]^2 dx + \epsilon^4 D_2\int_{-\infty}^{+\infty}\left[f\,\sin(\varphi_0) - \frac{1}{2\lambda_0}\cos(\varphi_0)\partial_x u^{(0)}\right]^2 dx.$$

The motion of the soliton is hence described by a Wiener process. The mean-square fluctuations of the solitary-wave position increases linearly with time according to the law

$$\langle[\Delta\xi(t)]^2\rangle=\left(\frac{D}{m\beta}\right)^2 t, \tag{35}$$

where $\Delta\xi(t)=\xi(t)-\xi_0$ and ξ_0 is the soliton position at $t=0$. In the weakly dissipative limit, the equations of motion for soliton position ξ and soliton velocity v are

$$\partial_t\xi=v,$$

$$m\beta\partial_t v=-2m\lambda\beta v+S(t) \tag{36}$$

where the real stochastic term $S(t)$ is δ correlated with the diffusion constant

$$D=\tfrac{4}{3}\epsilon^4\beta^3[D_1\cos^2(\varphi_0)+D_2\sin^2(\varphi_0)].$$

Equations (36) may be regarded as the Langevin equations of motion for a Brownian particle [25]. Assuming that at the time $t=0$ the soliton has the (deterministic) position $\xi=\xi_0$ and velocity $v=0$, it is straightforward to calculate the time evolution of the mean value and of the mean square fluctuation for the soliton position with standard techniques [25]. We find

$$\langle\xi(t)\rangle=\xi_0, \tag{37}$$

$$\langle[\Delta\xi(t)]^2\rangle=\left(\frac{D\tau}{m\beta}\right)^2\left[-\frac{3\tau}{2}-\frac{\tau}{2}\exp(-2t/\tau)+2\tau\exp(-t/\tau)+t\right] \tag{38}$$

where $\tau=1/2\lambda$ is the damping time associated with the viscous force. From Eq. (38) it follows that, for times t much shorter than the damping time τ, the behavior of $\langle[\Delta\xi(t)]^2\rangle$ may be approximated as

$$\langle[\Delta\xi(t)]^2\rangle\approx\frac{1}{3}\left(\frac{D}{m\beta}\right)^2 t^3,$$

i.e., in the first stage the variance of the soliton position increases with the third power of time t. Conversely, for times t larger than τ, one has

$$\langle[\Delta\xi(t)]^2\rangle\approx\left(\frac{D\tau}{m\beta}\right)^2 t,$$

i.e., the variance of the soliton position increases linearly with time as in the strongly dissipative limit. The conservative case is recovered from Eq. (36) by setting $\lambda=0$. In this case, the mean square fluctuation of the soliton position is given by

$$\langle[\Delta\xi(t)]^2\rangle=\frac{1}{3}\left(\frac{D}{m\beta}\right)^2 t^3, \tag{39}$$

which has a time-dependence typical for NLS solitons [23]. A comparison of Eqs. (35) and (39) clearly indicates that dissipation in the original system is able to reduce strongly the stochastic motion of solitons induced by external noise.

(4) *Soliton interactions*. The perturbative approach developed in Sec. III can be used to study interactions between two (or more) weakly overlapping solitary waves. The method is rather general and may be applied both to conservative and dissipative systems [17,26,27]. It consists of assuming a superposition of two individual solitary waves at positions ξ_1 and ξ_2 as a solution of the nonlinear wave equation (1). Because of the nonlinearity, this solution is not exact, but it represents a reasonable approximation provided that the solitary waves are weakly overlapping, i.e., for $1\ll\beta(\xi_2-\xi_1)$. The overlapping terms arising from the nonlinearity in the wave equation may then be regarded as perturbations to two wave equations for single solitary waves [26].

Using standard approximations to calculate overlapping integrals [26], in the strongly dissipative limit we obtain the following equations for the soliton positions ξ_1 and ξ_2:

$$\partial_t\xi_1=\frac{8\beta^3\delta_1\delta_2}{\lambda_0 m}\exp[-\beta(\xi_2-\xi_1)],$$

$$\partial_t\xi_2=-\frac{8\beta^3\delta_1\delta_2}{\lambda_0 m}\exp[-\beta(\xi_2-\xi_1)]. \tag{40}$$

The evolution equation for the soliton separation $\Delta\xi=\xi_2-\xi_1$ is hence

$$\partial_t\Delta\xi=-\frac{16\delta_1\delta_2\beta^3}{m\lambda_0}\exp(-\beta\Delta\xi). \tag{41}$$

From Eq. (41) it follows that, according to previous results [11,13,14], the two solitary waves attract each other for $\delta_1\delta_2=1$, i.e., if they have the same polarity, whereas they repel for $\delta_1\delta_2=-1$, i.e., if they have opposite polarity. In the former case, the soliton separation decreases with time, and the waves collide. However, the perturbative approach may not be used to study soliton dynamics during collision when their separation becomes comparable to β^{-1}. Previous numerical investigations on the two-wave states for the PGL equation have shown that, in the dissipative case, the two attracting solitary waves collapse into a single solitary wave with the same polarity, and that, during the collision, half of the field energy is transferred into dispersive waves which are rapidly attenuated with propagation [13]. The previous analysis can be easily extended to study soliton interactions in an extended chain of weakly overlapped solitary waves. Because of soliton interactions decreasing exponentially with soliton separation, only nearest-neighbor overlapping terms may be considered in the perturbation P. In this case, the coupled equations for the soliton positions ξ_n in the chain are given by

$$\partial_t\xi_n=\frac{8\beta^3}{\lambda_0 m}(\delta_n\delta_{n+1}\exp[-\beta(\xi_{n+1}-\xi_n)]-\delta_n\delta_{n-1}\exp[-\beta(\xi_n-\xi_{n-1})]), \tag{42}$$

where δ_n is the polarity of the nth solitary wave in the chain and $\xi_{n+1}>\xi_n$. According to Ref. [13], an infinitely extended chain of solitary waves with polarity alternation (i.e., $\delta_n\delta_{n+1}=-1$) and uniform spacing Δ ($\beta\Delta\gg1$) is a stable solution of Eqs. (42). This solution corresponds to exact periodic solutions of the PGL equation in the form of cnoidal waves [2,13].

Let us now consider interactions between two solitons in the weakly dissipative limit. In this case, from Eq. (23) we obtain the following dynamical equations:

$$m\partial_t^2\xi_1+2\lambda m\partial_t\xi_1-16\beta^3\delta_1\delta_2\ \exp[-\beta(\xi_2-\xi_1)]=0,$$
$$m\partial_t^2\xi_2+2\lambda m\partial_t\xi_2+16\beta^3\delta_1\delta_2\ \exp[-\beta(\xi_2-\xi_1)]=0. \tag{43}$$

From Eqs. (43) it follows that the soliton separation $\Delta\xi$ satisfies the equation

$$m\partial_t^2\Delta\xi+2\lambda m\partial_t\Delta\xi+32\beta^3\delta_1\delta_2\ \exp(-\beta\Delta\xi)=0, \tag{44}$$

which may be interpreted as the equation of motion of a particle of mass m in the potential

$$U(\Delta\xi)=-32\beta^2\delta_1\delta_2\ \exp(-\beta\Delta\xi)$$

under the action of a viscous force. As in the strongly dissipative case, the force is attractive if the solitary waves have the same polarity, and repulsive in the opposite case. Generalization of Eqs. (43) to a chain of weakly overlapped solitary waves is straightforward, and yields

$$m\partial_t^2\xi_n+2\lambda m\partial_t\xi_n-16\beta^3(\delta_n\delta_{n+1}\ \exp[-\beta(\xi_{n+1}-\xi_n)]$$
$$-\delta_n\delta_{n-1}\ \exp[-\beta(\xi_n-\xi_{n-1})])=0. \tag{45}$$

In the case of an infinite chain of solitons with polarity alternation ($\delta_n\delta_{n+1}=-1$), Eq. (45) show that the dynamics of solitary waves due to their weak interaction is described by a Toda lattice model with friction [28]. Because of the friction term, any internal oscillation of the lattice is damped and the periodic solutions of the PGL equation with uniform soliton spacing Δ ($\beta\Delta\gg 1$) are asymptotically reached. On the contrary, in the conservative limit, the friction term in Eq. (45) disappears, and collective excitations of the lattice are allowed. In particular, compression waves at a constant velocity may propagate through the lattice [28]; they are given by

$$r_n=-\frac{1}{\beta}\ \ln[1+\omega^2\ \mathrm{sech}(\Sigma n\pm\omega t)], \tag{46}$$

where $r_n=\xi_n-\xi_{n-1}-\Delta$ are the displacements of solitary waves in the chain from the periodic solution with uniform spacing Δ, $\omega=[16\beta^4\exp(-\beta\Delta)/m]^{1/2}\sinh\Sigma$, and Σ is an arbitrary parameter which determines the velocity of propagation and the compression factor of the collective wave.

V. CONCLUSIONS

In this paper the dynamics of parametrically excited solitary waves induced by perturbations has been theoretically investigated by using a direct perturbation approach. As the parametric term in the unperturbed equation breaks three of the four symmetries which are typical of the NLS equation, the effects of a generic perturbation on the soliton dynamics is simply to induce a motion of the soliton on a slow-time scale. It has been shown that the typical temporal scale over which the soliton motion occurs depends strongly on the conservative or dissipative character of the unperturbed problem. As a rather general rule, it turns out that dissipation in the unperturbed system is able to counteract the motion induced by perturbations. This behavior has been discussed in detail by considering some particular perturbations, which include external driving fields, higher-order correction terms in the unperturbed equation, noise sources, and soliton interactions. We envisage that such results may be of interest both from a fundamental point of view in understanding nonlinear dynamics of dissipative versus conservative solitons under the action of perturbations, and from an applicative point of view as well whenever robustness of soliton solutions is required.

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APPENDIX A: SINGULAR EIGENFUNCTION OF THE ADJOINT PROBLEM

In this appendix the singular eigenfunction w of the adjoint operator $\mathcal{L}^+$ is calculated. Let us first note that, because $\mathcal{L}_{11}$, $\mathcal{L}_{12}$, $\mathcal{L}_{21}$, and $\mathcal{L}_{22}$ are self-adjoint, one has

$$\mathcal{L}^+=\begin{pmatrix}\mathcal{L}_{11} & \mathcal{L}_{21}\\ \mathcal{L}_{12} & \mathcal{L}_{22}\end{pmatrix}=\begin{pmatrix}0 & \mathcal{L}_{21}\\ \mathcal{L}_{12} & -2\lambda_0\end{pmatrix}. \tag{A1}$$

Therefore the singular eigenfunction $w=w_1+iw_2$ satisfies the equations

$$\mathcal{L}_{21}w_2=0, \tag{A2}$$

$$\mathcal{L}_{12}w_1=2\lambda_0 w_2. \tag{A3}$$

Equation (A2) has the solution $w_2=\partial_x u^{(0)}$, whereas Eq. (A3) can be satisfied by setting $w_1=-2\lambda_0 f[\beta(x-\xi)]$, where the function $f(y)$ satisfies the following nonhomogeneous second-order linear equation

$$\left(-q^2+\frac{2}{\cosh^2 y}+\partial_y^2\right)f=h(y), \tag{A4}$$

with the boundary conditions $\lim_{y\to\pm\infty}f(y)=0$. In Eq. (A4) we set

$$h(y)=-\sqrt{2}\delta\,\frac{\sinh y}{\cosh^2 y} \tag{A5}$$

and

$$q=\left[\frac{-\vartheta-(\mu^2-\lambda_0^2)^{1/2}}{-\vartheta+(\mu^2-\lambda_0^2)^{1/2}}\right]^{1/2}. \tag{A6}$$

Note that the parameter q given by Eq. (A6) may vary in the interval [0,1] in order to ensure the stability of solitary waves. The solution of Eq. (A4) can be obtained by the method of variations of constants using a fundamental set of solutions $u_1(y), u_2(y)$ of the associated homogeneous equation

$$\left(-q^2+\frac{2}{\cosh^2 y}+\partial_y^2\right)u=0\,. \tag{A7}$$

A set of fundamental solutions to Eq. (A7) can be easily obtained by considering the variables

$$\omega=\tanh y, \tag{A8}$$

$$u(y)=F(y)\exp(\pm qy). \tag{A9}$$

Introduction of Eqs. (A8) and (A9) into Eq. (A7) yields the following equations for the unknown function $F(\omega)$:

$$(1-\omega^2)\partial_\omega^2 F-2(\omega\mp q)\partial_\omega F+2F=0, \tag{A10}$$

which admit the elementary solutions

$$F_{1,2}(\omega)=\omega\mp q \tag{A11}$$

The solutions given by Eq. (A11) correspond to the following functions for the original variables

$$u_1(y)=(\tanh y-q)\exp(qy), \tag{A12}$$

$$u_2(y)=(\tanh y+q)\exp(-qy). \tag{A13}$$

The Wronskian of u_1 and u_2 is given by

$$W=\begin{vmatrix} u_1 & u_2 \\ \partial_y u_1 & \partial_y u_2 \end{vmatrix}=-2q(1-q^2). \tag{A14}$$

Because $W\neq 0$ when q varies in the interval [0,1], solutions (A12) and (A13) are linearly independent. The solution $f(y)$ of the nonhomogeneous equation (A4) is hence given by

$$f(y)=V_1(y)u_1(y)+V_2(y)u_2(y), \tag{A15}$$

where

$$V_1(y)=\int_y^{+\infty}\frac{h(z)u_2(z)}{W}\,dz, \tag{A16}$$

$$V_2(y)=\int_{-\infty}^{y}\frac{h(z)u_1(z)}{W}\,dz, \tag{A17}$$

and the constants of integration have been chosen by imposing the vanishing of $f(y)$ for $y\to\pm\infty$. Using the symmetries $V_1(y)=V_2(-y)$ and $u_1(y)=-u_2(-y)$, we finally obtain

$$f(y)=\mathcal{T}(y)-\mathcal{T}(-y), \tag{A18}$$

where

$$\mathcal{T}(y)=\frac{\sqrt{2}\,\delta\,\exp(qy)(\tanh y-q)}{2q(1-q^2)}\times\int_{-\infty}^{-y}\frac{\exp(qz)\sinh z(\tanh z-q)}{\cosh^2 z}\,dz. \tag{A19}$$

APPENDIX B: SOLVABILITY CONDITION AT ORDER ϵ^2

In this appendix, the evolution equation (22) of the soliton position on the time scale T_1 is derived by imposing the solvability condition at $O(\epsilon^2)$. From Eqs. (13) and (21), it follows that the driving term $G^{(2)}$ is given by

$$G^{(2)}=a_0\partial_{T_2}\xi+i[a_1+a_2\partial_{T_1}\xi+a_3(\partial_{T_1}\xi)^2+a_4\partial_{T_1}^2\xi]+P[u^{(0)}\exp(i\varphi_0)]\exp(-i\varphi_0), \tag{B1}$$

where

$$a_0=\partial_x u^{(0)},$$

$$a_1=\frac{\lambda_1}{2\mu}u^{(0)}\left[1+\frac{u^{(0)^2}}{2\mu}\right],$$

$$a_2=-\lambda_1 f-\frac{\lambda_1}{2\mu}\partial_x u^{(0)}-\frac{\lambda_1}{\mu}fu^{(0)^2},$$

$$a_3=\partial_x f+f^2u^{(0)},$$

$$a_4=-f.$$

Because the eigenfunction w of the adjoint problem is imaginary and an odd function of $(x-\xi)$, the terms in $G^{(2)}$ proportional to a_0, a_1, and a_3 vanish when imposing the solvability condition (14). Therefore we obtain

$$\beta m\partial_{T_1}^2\xi+\lambda_1 k\beta\partial_{T_1}\xi+\mathrm{Im}\left[\int_{-\infty}^{+\infty}\partial_x u^{(0)}P\,\exp(-i\varphi_0)dx\right]=0, \tag{B2}$$

where the parameters m and k are given by

$$m=-\frac{1}{\beta}\int_{-\infty}^{+\infty}f\partial_x u^{(0)}dx=-\int_{-\infty}^{+\infty}f(y)\partial_y g(y)dy, \tag{B3}$$

$$k=m-\frac{\beta^2}{2\mu}\int_{-\infty}^{+\infty}[\partial_y g]^2dy-\frac{\beta^2}{\mu}\int_{-\infty}^{+\infty}f(y)g^2(y)\partial_y g(y)dy, \tag{B4}$$

and $g(y)=\sqrt{2}\delta/\cosh y$. The expression of the parameter k can be further simplified. In fact, the functions f and g satisfy the following equations:

$$(-q^2+\partial_y^2)f+g^2f=\partial_y g, \tag{B5}$$

$$(-1+\partial_y^2)g+g^3=0. \tag{B6}$$

Multiplying both sides of Eq. (B5) by $\partial_y g$ and integrating over the interval $(-\infty,+\infty)$, we obtain

$$\int_{-\infty}^{+\infty}(\partial_y g)^2 dy = q^2 m + \int_{-\infty}^{+\infty} g^2 f \partial_y g \; dy + \int_{-\infty}^{+\infty} \partial_y^2 f \partial_y g \; dy. \tag{B7}$$

The last integral on the right hand side in Eq. (B7) can be transformed by successive integrations by parts, using Eq. (B6), we obtain

$$\int_{-\infty}^{+\infty} \partial_y^2 f \partial_y g \; dy = -\int_{-\infty}^{+\infty} \partial_y f (g - g^3) dy$$
$$= \int_{-\infty}^{+\infty} f(\partial_y g - 3g^2 \partial_y g) dy. \tag{B8}$$

Insertion of Eq. (B8) into Eq. (B7) yields

$$\int_{-\infty}^{+\infty}(\partial_y g)^2 dy = (q^2-1)m - 2\int_{-\infty}^{+\infty} f g^2 \partial_y g \; dy. \tag{B9}$$

After observing that $\beta^2/\mu = 2/(1-q^2)$ from Eqs. (B4) and (B9), one has

$$k = 2m. \tag{B10}$$

Putting Eq. (B10) into Eq. (B2) we finally obtain Eq. (22) given in the text.

APPENDIX C: REGULAR PERTURBATION EXPANSION FOR CONSERVATIVE SOLITONS

In this appendix it is shown that for the PGL equation (1) in the conservative case there exists a general class of perturbations which does not induce any soliton motion. This means that a static (i.e., independent of time t) perturbation expansion of the solitary wave for the perturbed Eq. (7) can be constructed. The class of perturbations we consider has the form

$$P[u] = \mathcal{F}u, \tag{C1}$$

where $\mathcal{F}$ is a linear, real operator, independent of t, satisfying the property that, if g is an arbitrary even (odd) function of $(x-\xi)$, then $\mathcal{F}g$ is an odd (even) function of $(x-\xi)$. As an example, any linear combination with real coefficients of operators of the form $(x-\xi)^n \partial_x^m$, with m and n positive integers and $n+m$ odd, satisfies this property. From Eq. (18) it is clear that in the strongly dissipative limit the driving term on the right-hand side of the equation does not vanish, and perturbation (C1) induces a soliton motion on the time scale ϵ^{-2}. In the conservative limit, we look for a regular perturbation expansion of the solution $u(x,\epsilon)$ of Eq. (7) in the form

$$u(x,\epsilon) = [u^{(0)}(x) + \alpha u^{(1)}(x) + \alpha^2 u^{(2)}(x) + \cdots]\exp(i\varphi_0), \tag{C2}$$

where $\alpha = \epsilon^2$ and $\varphi_0 = \pi/4$. Substituting Eq. (C2) into Eq. (7), and collecting terms of the same order in α, a hierarchy of equations for successive corrections to u is obtained. At leading order, $O(\alpha^0)$, the solution $u^{(0)}$ is given by Eq. (10). The equations at higher orders have the form

$$\mathcal{L}u^{(k)} = -G^{(k)} \tag{C3}$$

where $\mathcal{L}$ is the linear operator given by Eq. (5) with $\lambda = 0$, and $G^{(k)}$ are driving terms which depend only on functions of previous approximations. Because $\mathcal{L}$ is singular, due to the Fredholm alternative theorem a bounded solution of Eq. (C3) exists if and only if the driving term in the equation is orthogonal to the singular eigenfunction $w = i\partial_x u^{(0)}$ of the adjoint problem. Therefore the perturbation expansion (C2) is consistent provided that the solvability conditions $(G^{(k)},w)=0$ are satisfied at any order k. This can be easily proved for $k=1$ and 2. In fact, for $k=1$ one has $G^{(1)} = -\mathcal{F}u^{(0)}$, which is a real function and therefore orthogonal to w. The solution at this order is imaginary and is given by $u^{(1)} = i\rho_1$, where ρ_1 is an odd function of $(x-\xi)$ which satisfies the equation $\mathcal{L}_{12}\rho_1 = -\mathcal{F}u^{(0)}$. At order $k=2$, we have $G^{(2)} = -2iu^{(0)}|u^{(1)}|^2 - \mathcal{F}u^{(1)}$, which is trivially an imaginary, even function of $(x-\xi)$. Hence it is orthogonal to w. The solution at $O(\alpha^2)$ is then given by $u^{(2)} = \rho_2$, where ρ_2 is an even function of $(x-\xi)$, and $\mathcal{L}_{12}\rho_2 = -\mathrm{Im}[G^2]$. By recursion, it is straightforward to show that, in general, $G^{(k)}(u^{(k)})$ is an imaginary (real) and even function of $(x-\xi)$ for k even, and a real (imaginary) odd function of $(x-\xi)$ if k is odd. In any case, the solvability conditions $(G^{(k)},w)=0$ are always satisfied. To prove this property, let us assume that it is valid up to the order $k=n-1$ (which, without loss of generality, we assume to be even), and let us show that the same property is still valid for $k=n$ and $k=n+1$. In fact, the driving term $G^{(k)}$ at order $k=n$ has the form

$$G^{(n)} = -i\sum_{m,s} u^{(m)}u^{(s)}u^{(n-s-m)*} - \mathcal{F}u^{(n-1)}, \tag{C4}$$

where the sum in Eq. (C4) is extended to all non-negative integers m and s such that $0 < s+m < n$. Because n is odd, it follows that each term in the sum is composed of the product of three functions of odd order, or of the product of two functions of even order and one of odd order. In both cases, it turns out that each term is a real, odd function of $(x-\xi)$. This is trivially true also for the last term in Eq. (C4). Hence $G^{(k)}$ is orthogonal to w. The solution at $O(\alpha^n)$ is then given by $u^{(n)} = i\rho_n$, where the real function ρ_n is determined as the solution of the equation $\mathcal{L}_{12}\rho_n = -G^{(n)}$. Furthermore, it is obvious that ρ_n has the same parity as $G^{(n)}$. For $k=n+1$, similar arguments can be used to show that $G^{(n+1)}$ is an imaginary and even function of $(x-\xi)$, and so it is orthogonal to w. The solution at $O(\alpha^{n+1})$ is then given by $u^{(n+1)} = \rho_{n+1}$, where the real function ρ_{n+1} has to be determined as the solution of the equation $\mathcal{L}_{21}\rho_{n+1} = -\mathrm{Im}[G^{(n+1)}]$, and has therefore the same parity as $G^{(n+1)}$. In conclusion, the solvability conditions at each order in the perturbation expansion are satisfied, and a hierarchy for successive approximations to u can indeed be constructed. The difficult problem of asymptotic convergence of the resulting perturbation series is left out in our discussion.

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